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Abstract

In this paper we show that we can replace the assumption of constant discount rate in the one-sector optimal growth model with the assumption of decreasing marginal impatience without losing major properties of the model. In particular, we show that the steady state exists, is unique, and has a saddle point property. All we need is to assume that the discount function is convex and has a uniformly bounded first-derivative.

JEL classification: O41, C61, C62
Key words: Solow equation, decreasing marginal impatience, saddle point, and bounded slope assumption.

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1 Introduction

Since the seminal papers of Koopmans (1960) and Uzawa (1968), economists have broadened the class of dynamic preferences to include recursive utility functions to tackle the problems[1] arising from the assumption of constant discount rate. Lucas and Stokey (1984) assumed increasing marginal impatience to ensure stability. Epstein (1987, pp.73-74) gave three reasons why employing the assumption of increasing marginal impatience is justified. Most studies, including the phase diagram analyses of the optimal growth models by Chang (1994, 2004) and Drugeon (1996), have focused on this increasing marginal impatience case.

The problems with decreasing marginal impatience are, as noted in Epstein (1983, p.140), that in deterministic models there exist many steady states and that some of them are locally unstable. The finding of "division of countries" by Magill and Nishimura (1984) is often cited as a reason to assume increasing marginal impatience. Specifically, Magill and Nishimura found that if the pure rate of time preference "decreases sufficiently rapidly" (p.281), then there exists a critical level of capital that separates the rich countries from the poor countries in such a way that the poor countries remain at subsistence, while the rich countries have permanent development.

[1] For example, Hicks (1965) argued that successive consumption units are supposed to be complementary, but an additively separable utility function implies that the marginal rate of substitution between lunch and dinner is independent of the type of breakfast one had that morning or expects to have the next morning. See Wan (1970, p.274). Additive separability also blurs the distinction between risk aversion and intertemporal substitution. See, for example, Duffie and Epstein (1992). Finally, additive separability has a peculiar long-run implication. Specifically, when there are heterogeneous agents, then, in the long run, the most patient consumer would own all the capital, while all other agents consume nothing and pay back their debts with all their labor income. See Becker (1980).

This finding, however, begs a question: What would happen to growth theory if the decrement in the pure rate of time preference were uniformly bounded?

Das (2003) demonstrated that a saddle-point steady state could be consistent with decreasing marginal impatience, using phase diagram analysis. Unfortunately, her analysis is incomplete and there is a technical error in the phase diagram analysis. These will be explained in the text. Most important, as pointed out in Lucas and Stokey (1984, p.169), the purpose of studying recursive utility is to see how far we can relax the assumption of convenience, namely the assumption of constant discount rate, without losing convenience. Professor Das offered no economic interpretation, other than mathematical necessity, for the required stability condition. Therefore, it is not clear what is the value added of Das (2003) to the literature. As we will show later, her stability assumption would actually create some inconvenience.

This paper studies the existence, the uniqueness, and the stability issue of one-sector optimal growth with decreasing marginal impatience, following the phase diagram analysis of Chang (1994). To accomplish our goal, we need only to impose an additional restriction on the discount function, while keeping the usual assumptions of preferences and technologies employed in the constant discount rate case. Specifically, we need to assume that the discount function is strictly convex and that the slope of the discount function at zero consumption is bounded from below. Henceforth, the latter condition is referred to as the "bounded slope" assumption. The convexity and the bounded slope assumption of the discount function together imply that the slope of the discount function is uniformly bounded. A constant discount function can obviously be regarded as its limiting case. In so doing, we have extended the scope of growth theory without creating unnecessary inconveniences.

Under this bounded slope assumption, we show that the steady state of optimal growth with decreasing marginal impatience exists and is unique. Furthermore, the steady state has the usual saddle point property. Recall that, in the constant discount rate case, the curve corresponding to the steady state of consumption is a vertical line in the phase plane. We show that, in the decreasing marginal impatience case, the curve corresponding to the steady state of consumption is obtained from "bending" the upper and lower parts of a vertical line rightward so that it is upward sloping in the upper part and downward sloping in the lower part. The upper part of the curve is more like a bell-shaped curve than a C-shaped curve because it is asymptotic to another vertical line. This steady state retains all qualitative properties of the steady state in the constant discount rate case.

The differences between the case of decreasing marginal impatience and the case of increasing marginal impatience are in the derivatives of the discount function. Borrowing the results from Chang (1994), we show that the stability results in those two cases are "mirror images" of each other.

Since the decreasing marginal impatience case is a "mirror image" of the increasing marginal impatience case and conversely, and since the constant discount rate case is the limit of either case, we conclude the paper with a unifying presentation that contains all three cases in a single diagram. It makes clear the effects of monotonic marginal impatience, in comparison with a constant discount rate, on the steady state consumption and steady state capital. It also makes a clear statement about the stability analysis of monotonic marginal impatience and its saddle point property.

2 The Model

The model of optimal growth with decreasing marginal impatience is similar to the model of increasing marginal impatience of Chang (1994). The major difference is in the presentation of the bounded slope assumption.

The law of motion is the standard Solow equation

$$\dot{k} = f(k) - c - nk. \tag{1}$$

The per capita production function $f(k)$ is assumed to be of class C^2 (twice continuously differentiable), strictly increasing, strictly concave, satisfying the Inada conditions:

$$f(0) = 0, \ \lim_{k \to 0} f'(k) = \infty, \text{ and } \lim_{k \to \infty} f'(k) = 0.$$

The objective function is

$$\int_0^\infty D(t)\, U(c_t)\, dt, \tag{2}$$

where $U(c)$ is the instantaneous utility function, which is of class C^2, strictly increasing and strictly concave in c and

$$D(t) = \exp\left\{ - \int_0^t \delta(c_s)\, ds \right\}, \ \delta(c_s) > 0, \tag{3}$$

is the discount factor at time t. We shall refer to $\delta(c)$ as the (instantaneous) discount function.

Clearly, $D(t)$, which assumes values in $(0, 1]$, depends on the underlying consumption path $\{c_s : s \leq t\}$, and is *decreasing in time* because

$$D'(t) = -D(t)\, \delta(c_t) < 0.$$

By definition, $D(0) = 1$ and $D'(0) = -\delta(c_0) < 0$. In the classic case of a constant discount rate, i.e., $\delta(c_t) = \delta$, a constant, we have $D(t) = e^{-\delta t}$ with $D'(t) = -D(t)\delta$.

What distinguishes the case of increasing marginal impatience from the case of decreasing marginal impatience is the functional structure of $\delta(c)$. By increasing marginal impatience we mean $\delta'(c) > 0$, and by decreasing marginal impatience we mean $\delta'(c) < 0$. In the case of decreasing marginal impatience, we also assume $\delta(0) = b > 0$ and $\delta''(c) > 0$ so that $\delta(c)$ is defined for all $c \geq 0$. We assume that $\delta(c)$ is asymptotic to the horizontal axis $\delta = 0$ so that

$$0 < \delta(c) \leq b. \tag{4}$$

Then the optimal growth problem with decreasing marginal impatience is formulated as

$$\max_{\{c_t\}} (2), \text{ s.t. } (1). \tag{5}$$

It is standard to verify that the value function of (5), in current value form, is independent of the initial time, and depends only on the initial capital-labor ratio. Hence, we denote it by $J(k)$. For the moment, we assume $U(c) \geq 0$ so that $J(k) \geq 0$. The Bellman equation[2] for problem (5) is

$$0 = \max_{c} \left\{ U(c) - \delta(c) J(k) + [f(k) - c - nk] J'(k) \right\}. \tag{6}$$

The first-order condition of (6) is

$$U'(c) - \delta'(c) J(k) - J'(k) = 0, \tag{7}$$

[2] The method of deriving the Bellman equation, the costate equation, and the Euler equation is standard in the literature. See, for example, Chang (1994, 2004). The stochastic version of the Bellman equation for this class of objective functions is derived in Krylov (1980, p.25) or Chang (2004).

and the second order sufficient condition is

$$U^{''}(c) - \delta^{''}(c) J(k) < 0. \tag{8}$$

The second order condition (8) is clearly satisfied because $J(k) \geq 0$.

Let the costate variable be $p = J^{'}(k)$, which is the shadow price of k. This shadow price

$$p = U^{'}(c) - \delta^{'}(c) J(k)$$

is clearly positive since $J(k) \geq 0$. Then, the costate equation is

$$\dot{p} = -p \left[f^{'}(k) - n - \delta(c) \right], \tag{9}$$

which corresponds to equation (12) in Das (2003). Substituting

$$p = U^{'}(c) - \delta^{'}(c) J(k)$$

into (9), we have the Euler equation

$$b(k, c) \dot{c} = f^{'}(k) - n - \delta(c) - \delta^{'}(c) [f(k) - c - nk], \tag{10}$$

where

$$b(k, c) = -\frac{U^{''}(c) - \delta^{''}(c) J(k)}{U^{\prime}(c) - \delta^{'}(c) J(k)}.$$

Equation (10) corresponds to equation (15) [after applying her equation (23)] in Das (2003).

The sign of the numerator of $b(k, c)$ is the same as the second order sufficient condition (8), which is negative. The denominator of $b(k, c)$ is the shadow price p, which is positive. Therefore, $b(k, c) > 0$ and the Euler equation (10) is nondegenerate. Notice that the sign of the change in consumption (the sign of $\dot{c}$) is independent of the utility function $U(c)$.

3 Stability Analysis

To study the phase diagram analysis, we focus on the pair of equations (1) and (10) in the (k, c)-plane. Let $\bar{k}$ be the maximal sustainable capital-labor ratio, i.e., $f\left(\bar{k}\right) = n\bar{k}$. We can therefore restrict our discussion to the compact interval $\left[0, \bar{k}\right]$.

As discussed in the introductory section, a uniform bound on the slope of the discount function may be a candidate for stability of optimal growth with decreasing marginal impatience. Since $\delta(c)$ is convex, a lower bound on $\delta'(0)$, or an upper bound on $\left|\delta'(0)\right|$, will be sufficient for a uniformly bounded $\delta'(c)$.

Formally, let

$$k_1 = \left(f'\right)^{-1}(n+b) = \left(f'\right)^{-1}(n+\delta(0)),$$

and

$$k_2 = \left(f'\right)^{-1}(n) = \left(f'\right)^{-1}\left(n + \lim_{c\to\infty}\delta(c)\right).$$

Since $b > 0$, we have $k_1 < k_2$.

Bounded slope assumption. Assume

$$-\delta'(0) \le \left(\frac{1}{b}\right) \min_{k\in[k_1,k_2]}\left[-f''(k)\right].$$

The interval $[k_1, k_2]$ is a compact set and therefore the minimum exists and is finite. The upper bound is given by the exogenous parameters n, b, and the production function $f(k)$, and nothing else. In other words, it is the population growth rate, the production technology and the bounds of $\delta(c)$ that determine the lower bound for $\delta'(0)$. In particular, if $f''(k)$ is monotonic, then $\min_{k\in[k_1,k_2]}\left[-f''(k)\right]$ is simplified to $\min\left\{-f''(k_1), -f''(k_2)\right\}$.

3.1 Steady State

The steady state, if it exists, is denoted by (k_d, c_d). By definition, (k_d, c_d) satisfies $\dot{k} = \dot{c} = 0$, i.e.,

$$f(k) - c - nk = 0, \tag{11}$$

and

$$f'(k) - n - \delta(c) = 0. \tag{12}$$

The curve defined by (11), denoted by L_1, is of inverted "U" shape with k-axis intercepts 0 and $\bar{k}$, which is the same as the constant discount rate case (and increasing marginal impatience case as well).

The curve defined by (12), denoted by L_2, is upward sloping, i.e.,

$$\frac{dc}{dk} = \frac{f''(k)}{\delta'(c)} > 0,$$

because $f''(k) < 0$ and $\delta'(c) < 0$. The k-axis intercept ($c = 0$) of the curve L_2 is k_1. This means that L_2 is not defined for $k < k_1$ because $\delta(c) \leq b$. For $k \geq k_2$, we have $f'(k) - n \leq 0$, which implies that $\delta(c) \leq 0$ if (12) is satisfied. In other words, L_2 is not defined on $k \geq k_2$ either. In summary, L_2 is defined only on $[k_1, k_2)$, and on which $0 < f'(k) - n \leq b$.

As k approaches k_2 from the left, $f'(k) \to n$. Along the curve L_2, $\delta(c) = f'(k) - n \to 0$, and therefore, $c \to \infty$. This observation says that L_2 is asymptotic to the vertical line $k = k_2$. This asymptotic property implies that the curve L_2 must cross L_1 at least once, i.e., *a* steady state always exists.

To show the uniqueness of the steady state, it suffices to show that the two curves L_1 and L_2 cross each other only once. Since $0 < f'(k) - n \leq b$

on $[k_1, k_2)$, we have, for any c,

$$-\delta^{'}(c)\left[f^{'}(k)-n\right] \leq -\delta^{'}(c)\, b, \text{ for all } k \in [k_1, k_2).$$

The above inequality remains valid for $k = k_2$ because $f^{'}(k_2) = n$. For *any* $c > 0$, we have $\delta^{'}(0) < \delta^{'}(c) < 0$ because $\delta^{''}(c) > 0$. It follows that $-\delta^{'}(c)\, b < -\delta^{'}(0)\, b$. Then the bounded slope assumption implies that, for *any* $c > 0$,

$$-\delta^{'}(c)\left[f^{'}(k)-n\right] < -f^{''}(k), \text{ for all } k \in [k_1, k_2]. \tag{13}$$

Since (13) is valid for all $c > 0$, it is valid when $c = \delta^{-1}\left(f^{'}(k)-n\right)$ with $k \in [k_1, k_2)$, i.e., as we move along L_2. In this case, we have

$$f^{'}(k)-n < \frac{f^{''}(k)}{\delta^{'}\left(\delta^{-1}\left(f'(k)-n\right)\right)}.$$

This inequality says that the slope of the curve L_2, $f^{''}(k)/\delta^{'}(c)$, is strictly greater than the slope of the curve L_1, $f^{'}(k)-n$, at any point in the interval $[k_1, k_2)$. This makes the second crossing (including tangency) of the two curves impossible.

The unique intersection of the two curves L_1 and L_2 defines the steady state (k_d, c_d), where $k_d \in (k_1, k_2)$. That is, the steady state is in the increasing section of the L_1 curve. See Figure 1. In summary, we have

Proposition 1 *Under the bounded slope assumption, the steady state of optimal growth with decreasing marginal impatience exists and is unique.*

Some comparative dynamics can easily be obtained. A decrease in the population growth rate, n, "expands" the curve L_1 and shifts the curve L_2

to the right as shown in Figure 2. Steady state consumption and capital are unambiguously increased. This result is quite intuitive because there are simply fewer people to feed and to share the existing capital. Similarly, a Hicks-neutral technical progress, i.e., the new production technology is characterized by $Af(k)$, $A > 1$, would also expand the curve L_1 and shifts the curve L_2 to the right as shown in Figure 2. Again, steady state consumption and capital are unambiguously increased. This is also quite intuitive because such a technological change represents a scale effect on production and hence on consumption. In both cases, the results resemble the comparative dynamics of the optimal growth model with a constant discount rate.

3.2 Phase Diagram

Let the curve defined by $\dot{c} = 0$ be $L_3 : R(k, c) = 0$, where

$$R(k, c) = f'(k) - n - \delta(c) - \delta'(c)[f(k) - c - nk]. \tag{14}$$

The location of the curve L_3 can be determined as follows. First, we recognize that L_1 and L_2 divide the first quadrant of the (k, c)-plane into four sectors: A, B, C, and D. See Figure 3. In sector B we have $R(k, c) < 0$, because it is the region above the curve L_1 (i.e., $f(k) - c - nk < 0$) and below (or to the right of) the curve L_2 (i.e., $f'(k) - n - \delta(c) < 0$). Therefore, the curve $L_3 : R(k, c) = 0$ cannot lie in this sector. Similarly, in sector C we have $R(k, c) > 0$, because it is the region below the curve L_1 (i.e., $f(k) - c - nk > 0$) and above (or to the left of) the curve L_2 (i.e., $f'(k) - n - \delta(c) > 0$). Again, the curve $L_3 : R(k, c) = 0$ cannot lie in this sector. Therefore, the curve $L_3 : R(k, c) = 0$ must be located in sector A (above L_1 and L_2) and sector D (below L_1 and L_2). When L_3 is in sector A, we have $k < k_2$. Similarly, when L_3 is in sector D, we have $k > k_1$.

Next, we examine the behavior of the curve L_3 in the strip S defined by

$$S = \{(k, c) : k \in [k_1, k_2]\}.$$

For all $(k, c) \in S$, we have

$$R_k(k, c) = f''(k) - \delta'(c) \left[f'(k) - n \right] < 0,$$

using (13). Applying the implicit function theorem to $R(k, c) = 0$, k *can be written as a function of* c, which has a derivative

$$\frac{dk}{dc} = -\frac{R_c}{R_k} = \frac{\delta''(c)[f(k) - c - nk]}{f''(k) - \delta'(c)[f'(k) - n]}. \tag{15}$$

[Equation (15) corresponds to equation (A.23) in Das (2003) that computes dc/dk.] Note that the implicit function theorem applies to all points of $R(k, c) = 0$ in S. Inequality (13) implies that the denominator of (15) is negative on S so that the sign of dk/dc depends only on the sign of the numerator of (15). Since $\delta''(c) > 0$, dk/dc has a sign opposite of $\dot{k}$.

If (k, c) lies above L_1, i.e., $f(k) - c - nk < 0$, then $\dot{k} < 0$ and hence k is increasing in c. Similarly, if (k, c) lies below L_1, i.e., $f(k) - c - nk > 0$, then $\dot{k} > 0$ and hence k is decreasing in c. At (k_d, c_d), dk/dc equals zero. Thus, the curve L_3 in the strip S can be obtained from "bending" the upper and lower part of the vertical line $k = k_d$ rightward so that it is upward sloping in the upper part and downward sloping in the lower part. See Figure 4. Using the expression of a "C-shaped" curve for L_3 would be misleading because the upper part of L_3 is asymptotic to another vertical line $k = k_2$. In fact, the upper part of L_3 stays in the strip $\{(k, c) : k_d \leq k \leq k_2\}$, which is smaller than S.

If the lower portion of L_3 extends to the region $\left[k_2, \bar{k}\right]$, the curve is still downward sloping. This is because in this region, $f'(k) \leq n$, which implies $R_k(k, c) < 0$, and the implicit function theorem is still applicable. In any event, the lower part of L_3 stays in the strip $\left\{(k, c) : k_d \leq k \leq \bar{k}\right\}$.

Now we are ready to determine the vertical arrows of the phase diagram. Since $R_k(k, c) < 0$ on $[k_1, k_2]$, $R(k, c) > 0$ and hence $\dot{c} > 0$ in the region to the left of L_3. Similarly, we have $R(k, c) < 0$ and hence $\dot{c} < 0$ in the region to the right of L_3. In summary, the vertical arrows are as follows:

$$\begin{aligned} \dot{c} &< 0 \text{ if } (k, c) \text{ is to the right of } L_3; \\ \dot{c} &> 0 \text{ if } (k, c) \text{ is to the left of } L_3. \end{aligned}$$

The horizontal arrows are the same as the constant discount rate case, i.e.,

$$\begin{aligned} \dot{k} &< 0 \text{ if } (k, c) \text{ is above the curve } L_1; \\ \dot{k} &> 0 \text{ if } (k, c) \text{ is below the curve } L_1. \end{aligned}$$

Combining all arrows, a complete phase diagram is shown in Figure 4.

Proposition 2 *The steady state of the optimal growth model with decreasing marginal impatience is a saddle point.*

3.3 The case of $U(c) < 0$

If $U(c) < 0$, then $J(k) < 0$. In this case we need to *assume* that the second order sufficient condition (8) is valid and that

$$-\frac{\delta''(c)}{\delta'(c)} \geq -\frac{U''(c)}{U'(c)}. \tag{16}$$

The inequality (16) says that the degree of convexity of $\delta(c)$ is greater than the degree of concavity of $U(c)$. Then the shadow price of capital-labor ratio satisfies

$$\begin{aligned} p &= U'(c) - \delta'(c) J(k) = U'(c) \left[1 - \frac{\delta'(c)}{U'(c)} J(k) \right] \\ &\geq U'(c) \left[1 - \frac{\delta''(c)}{U''(c)} J(k) \right] = \frac{U'(c)}{U''(c)} \left[U''(c) - \delta''(c) J(k) \right] > 0, \end{aligned}$$

the last inequality is obtained from (8). It is then straightforward to verify that the properties such as the existence, the uniqueness, and the saddle point property of the steady state remain valid.

It is interesting to point out that, in the case of increasing marginal impatience, (8) and $p > 0$ are automatic if $U(c) < 0$. But assumptions (8) and (16) are required for $p > 0$ if $U(c) \geq 0$. In that case, assumption (16) simply says that $\delta(c)$ is more concave than $U(c)$. See Chang (1994) and Drugeon (1996) for details.

3.4 Comments on Das (2003)

For stability analysis, Professor Das assumed the following inequality

$$-f''(k) > -\delta'(f(k) - nk) \left[f'(k) - n \right], \text{ for all } k \in (0, k_2], \tag{17}$$

without offering an economic interpretation. Even though this inequality resembles (13), they are quite different. The simplest explanation is that inequality (17) is a one-dimensional condition (in k alone), but (13) is a two-dimensional inequality (in k and c). To elaborate, let

$$g(k, c) = f''(k) - \delta'(c) \left[f'(k) - n \right]. \tag{18}$$

Inequality (17) is obtained by substituting $c = c(k) = f(k) - nk$ into (18) so that (17) can be written as $g(k, c(k)) < 0$. In so doing, the inequality (17) is a condition along, and at best applicable to some neighborhood of, $L_1 : c = f(k) - nk$, not to the entire first quadrant of the (k, c)-plane. As such, it is a *local* condition that cannot be used to analyze the shape of the L_3 curve located away from the L_1 curve in the (k, c)-plane. Therefore, Professor Das's description of the L_3 curve, and therefore her stability analysis, is incomplete.

Furthermore, the domain of (17) is $(0, k_2]$. As we noted earlier, the curve L_2 is defined only on $[k_1, k_2)$. As $k \to 0$, we have $f'(k) \to \infty$ and inequality (17) may fail. Similarly, inequality (17) may fail if $\delta'(0)$ is unbounded. Imposing conditions on $f''(k)$, as $k \to 0$, to tackle the problems just mentioned would not be in the direction of relaxing the assumptions of convenience.

In addition, Das (2003) studied only the case with $U(c) > 0$. In contrast, we extend the stability results to the growth models with decreasing marginal impatience in which $U(c) < 0$.

The technical error in Das (2003) is in the stability analysis. The assumption (17) ensures that inequality (13) is valid at the steady state (k_d, c_d), i.e.,

$$f''(k_d) - \delta'(c_d)\left[f'(k_d) - n\right] < 0 \tag{19}$$

so that dk/dc is zero at (k_d, c_d). What Professor Das failed to recognize is that the function $g(k, c)$ is continuous in the (k, c)-plane. By continuity, if $g(k_d, c_d) > 0$, then we must have $g(k, c) > 0$ in some neighborhood of (k_d, c_d). That is,

$$f''(k) - \delta'(c)\left[f'(k) - n\right] < 0$$

in that neighborhood of (k_d, c_d). Therefore it is impossible for the denominator of (15) to change signs in that same neighborhood of (k_d, c_d). Instead, Professor Das argued that the sign of the denominator of (15) is ambiguous. This error led Professor Das to draw the conclusion that the curve L_3 may be of "S" shape as represented by her Figure 3. Most important, this inescapable error is completely within Professor Das's own model and framework.

4 All Cases Considered

It would be useful and instructive to compare all cases (decreasing marginal impatience, constant marginal impatience, and increasing marginal impatience) in a single framework and in the same diagram. To this end, we borrow from Chang (1994) the results of the case of increasing marginal impatience in which $\delta^{'}(c) > 0$ and $\delta^{''}(c) < 0$.

The curve defined by (12), denoted by L_4, is downward sloping, $dc/dk < 0$, due to $f^{''}(k) < 0$ and $\delta^{'}(c) > 0$. Its k-axis intercept is

$$k_3 = \left(f^{'}\right)^{-1}(n + \delta(0)) < \left(f^{'}\right)^{-1}(n) = k_2.$$

Therefore, the curve L_4 intersects the curve L_1 at its upward sloping portion and the intersection is unique. Denote it by (k_i, c_i). See Figure 5. Notice that, unlike the decreasing marginal impatience case, the steady state (k_i, c_i) is uniquely determined without a bounded slope assumption.

The curves L_1 and L_4 partition the first quadrant of the (k, c)-plane into four sectors. It can be verified that the curve $L_5 : R(k, c) = 0$ is located in sector B that is above the curve L_1 and to the right of L_4, and in sector C that is below the curve L_1 and to the left of the curve L_4. See Figure 6.

The phase diagram can be obtained similarly. Let the curve defined by $\dot{c} = 0$ be $L_5 : R(k, c) = 0$, where $R(k, c)$ is defined in (14). It is shown in Chang (1994), using the strict concavity of $\delta(c)$, that $R(k, c) < 0$ on $\left[k_2, \bar{k}\right]$ for any c. Therefore, the location of the curve L_5 is in the strip

$$\{(k, c) : k \in [0, k_2]\}.$$

In this strip, $f''(k) - \delta'(c)\left[f'(k) - n\right] < 0$, i.e., (13) is always satisfied. Notice that we do not need to assume a bounded slope condition in this case. The phase diagram of optimal growth with increasing marginal impatience is drawn in Figure 7.

The case of constant impatience $\delta(c) = \delta$ is well-known. See, for example, Intriligator (1971). In that case, the curve associated with $\dot{c} = 0$ (or $f'(k) - n - \delta = 0$) is a vertical line, $k = k_0$, where

$$k_0 = f^{-1}(n + \delta).$$

The phase diagram is reproduced in Figure 8.

To make meaningful connections among the three cases, we have to relate the constant δ in the constant discount rate case to $\delta(0)$ in the other two cases. In the increasing marginal impatience case, we assume $\delta(0) = \delta$ (i.e., $k_0 = k_3$) so that $\delta(c) > \delta$ for all $c > 0$. In the decreasing marginal impatience case, we assume $\delta(0) = b = \delta$ (i.e., $k_0 = k_1$) so that $\delta(c) < \delta$ for all $c > 0$. Treated this way, the constant discount rate becomes the limiting case of both decreasing and increasing marginal impatience growth models. Figure 9 shows the relative position of the steady states among the three cases. Figure 10 shows that the effect of changing from constant marginal impatience to monotonic marginal impatience is simply to shift and bend the vertical line $k = k_0$; the rest of the stability analysis essentially remains unchanged.

5 Concluding Remarks

In this paper we show that we can replace the assumption of constant discount rate in the one-sector growth model with decreasing marginal impatience without losing any major properties of the model. In particular, the major properties such as the existence, the uniqueness, and the saddle point property of the steady state remain valid. All we need is to assume that the discount function is convex and has a uniformly bounded first-derivative.

We also show that the phase diagram analysis of the optimal growth with decreasing marginal impatience is "symmetric" to the phase diagram analysis of the optimal growth with increasing marginal impatience, and that the constant discount rate case can be regarded as the limiting case of either model. It suggests that there is no longer any excuse to restrict ourselves to the assumption of constant discount rate, at least for the continuous-time one-sector optimal growth model.

The bounded slope assumption is closely related to the work of Magill and Nishimura (1984). However, it is not always true that continuous-time results would automatically imply discrete-time ones, nor the converse. See Chang (1988, 2004) for discussion. The effects of the bounded slope assumption on the discrete time models remain to be investigated. This is for future research.

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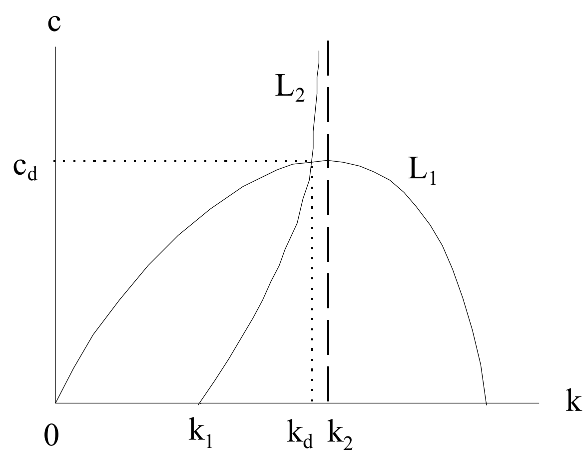

Figure 1: Existence and uniqueness of the steady state

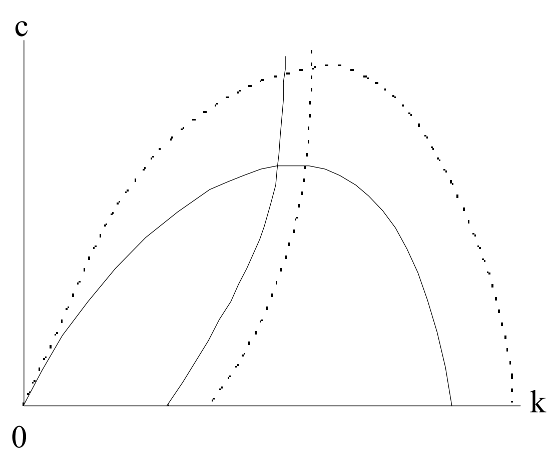

Figure 2: Comparative Dynamics

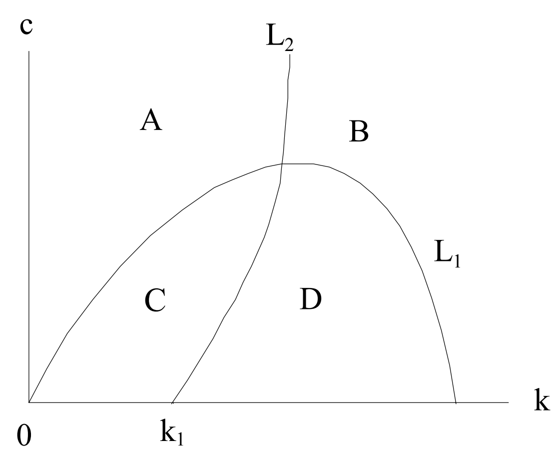

Figure 3: The curve L_3 lies in region A and region D.

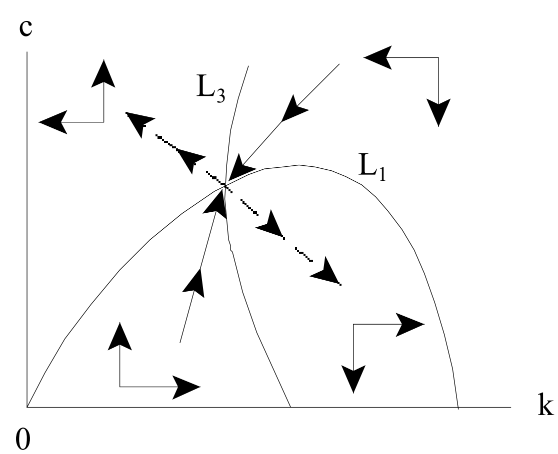

Figure 4: The steady state is a saddle point

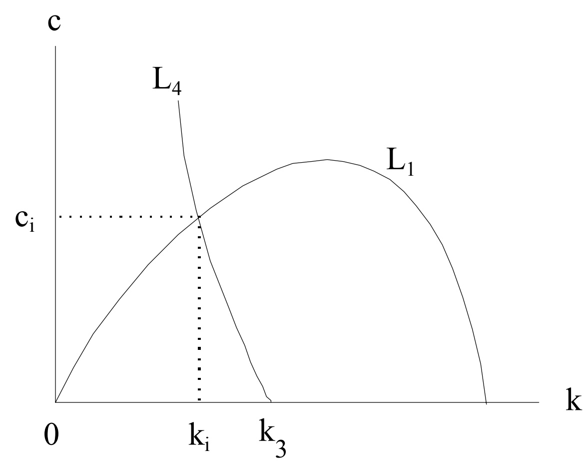

Figure 5: Existence and uniqueness of the steady state (increasing marginal impatience)

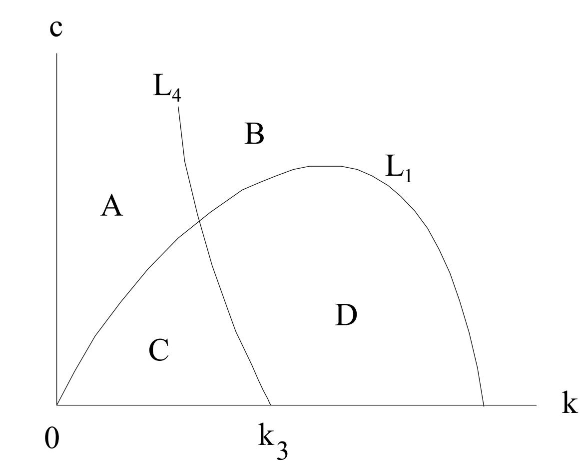

Figure 6: The curve L_5 lies in sector B and C

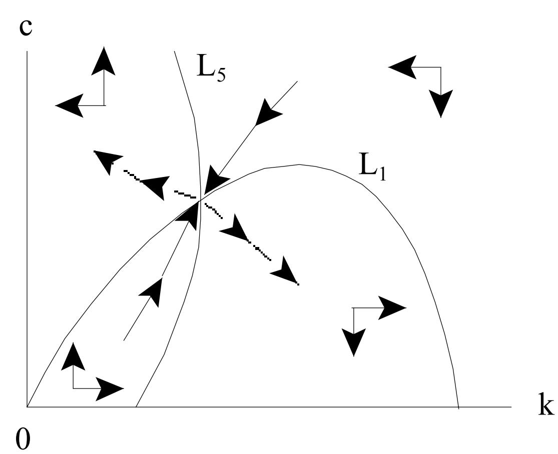

Figure 7: The steady state is a saddle point (increasing marginal impatience)

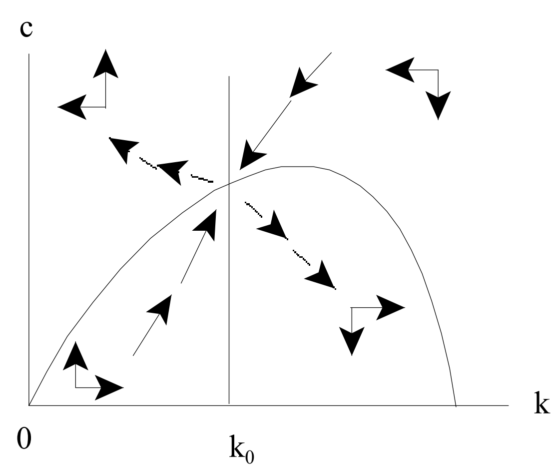

Figure 8: The phase diagram of constant discount rate case

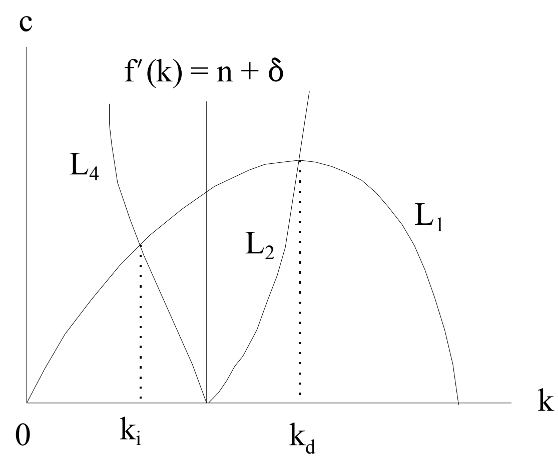

Figure 9: Comparison of the location of the steady states

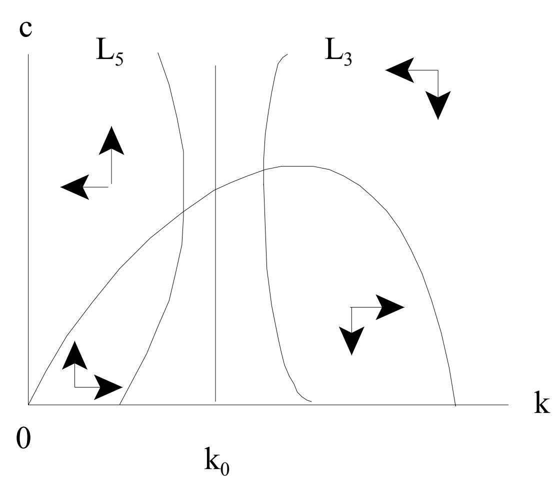

Figure 10: To change from constant marginal impatience to non-constant one is to bend the vertical line $k = k_0$